Isopycnal mixing and backscatter parameterizations

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Parameterization at coarse resolution

Advective transport

(Gent and McWilliams, 1990)

$$\boldsymbol{u}_* = -\partial_z(\kappa_{\mathrm{GM}}\boldsymbol{s}),$$
$$w_* = \nabla_h \cdot (\kappa_{\mathrm{GM}}\boldsymbol{s})$$

$$\boldsymbol{s} = -\nabla_h\rho/\partial_z\rho$$

$$A_{\mathrm{GM}} = \kappa_{\mathrm{GM}} \begin{bmatrix} 0 & 0 & -s_x \\ 0 & 0 & -s_y \\ s_x & s_y & 0 \end{bmatrix}$$

- adiabatic slumping of isopycnals

→ eddy-induced transport velocity

→ GM coefficient (“thickness diffusivity”)

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Diffusive mixing

(Redi, 1982)

$$\nabla \cdot (S_{\mathrm{R}} \nabla \overline{c})$$

where

$$S_{\mathrm{R}} = \kappa_{\mathrm{R}} \begin{bmatrix} 1 & 0 & s_x \\ 0 & 1 & s_y \\ s_x & s_y & |\boldsymbol{s}|^2 \end{bmatrix}$$

- eddy stirring enhances dissipation of tracer variance

→ downgradient diffusion of tracers on isopycnals

→ Redi coefficient (isopycnal tracer diffusivity)

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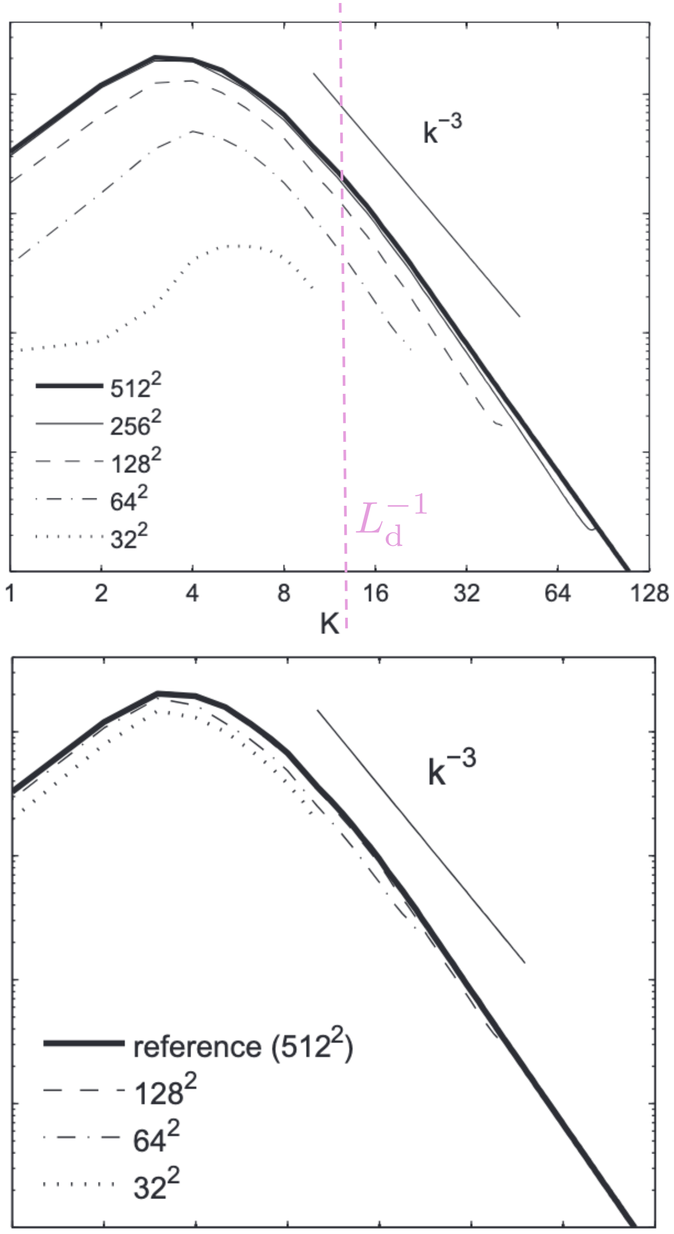
$$\partial_t \overline{c} + \overline{\boldsymbol{v}} \cdot \nabla \overline{c} - \nabla \cdot (A_{\mathrm{GM}} \nabla \overline{c}) = \nabla \cdot (S_{\mathrm{R}} \nabla \overline{c}) + \mathrm{source/sink}$$

Parameterization at eddy-permitting resolution

- GM-Redi designed for **non-eddying** resolutions

- Mesoscale turbulence generated by baroclinic instability **partially resolved**
    - o resolution near deformation radius
      → dissipation cuts off energy source for inverse cascade

- **Energy backscatter** to re-energize partially resolved eddies using negative viscosity



Jansen and Held (2014)

Parameterization at eddy-permitting resolution

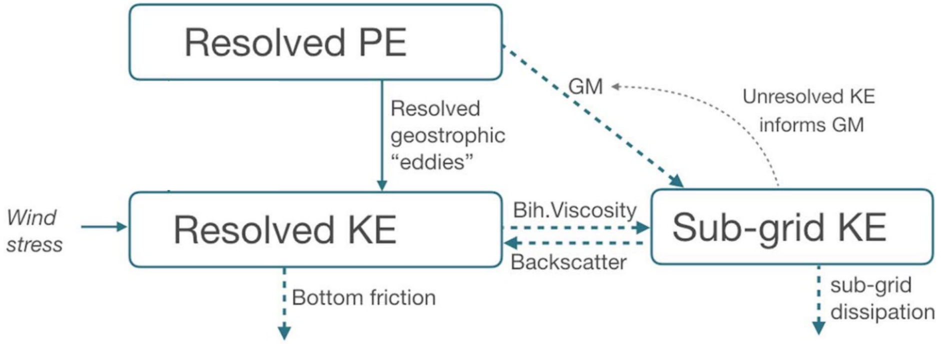
Backscatter in more realistic set-ups:

$$\partial_t \boldsymbol{u} + (f+\zeta)\widehat{\boldsymbol{z}} \times \boldsymbol{u} + \nabla(K+M) = \frac{1}{\rho_0}\partial_z \boldsymbol{\tau} - \underbrace{\boxed{\nabla \cdot \left[\nu_4 \nabla(\nabla^2 \boldsymbol{u})\right]}}_{\text{Dissipation}} + \overbrace{\boxed{\nu_2 \nabla^2 \boldsymbol{u}}}^{\text{Backscatter}}$$

$$\frac{\partial e}{\partial t} = \dot{e} = \dot{e}_{\mathrm{GM}} + \dot{e}_{\mathrm{Smag}} - \dot{e}_{\mathrm{BScat}} - \dot{e}_{\mathrm{diss}} - \dot{e}_{\mathrm{adv}}$$

$$\nu_2 = c\sqrt{2e}L_{\mathrm{mix}}, \quad e = e(x,y,t)$$

- negative viscosity from prognostic **subgrid KE budget**



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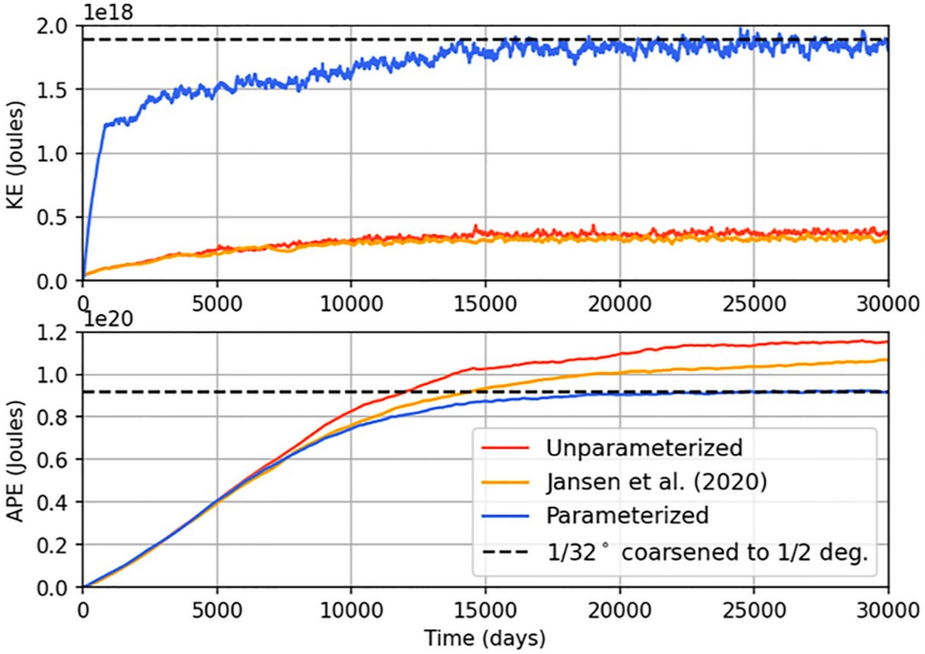
Backscatter **with no GM** can get both the kinetic energy *and* the stratification to match a hi-res truth
(Yankovsky et al. 2024)



Yankovsky et al. (2024)

Hypothesis & goal

Hypothesis:

That backscatter induces sufficient isopycnal mixing

Goal:

Measure mixing in model simulations

Experimental set-up

- Adiabatic primitive equations on sphere
- MOM6, NeverWorld2 configuration
  (Marques et al. 2022)

→ 15 isopycnal layers

→ constant wind stress at surface

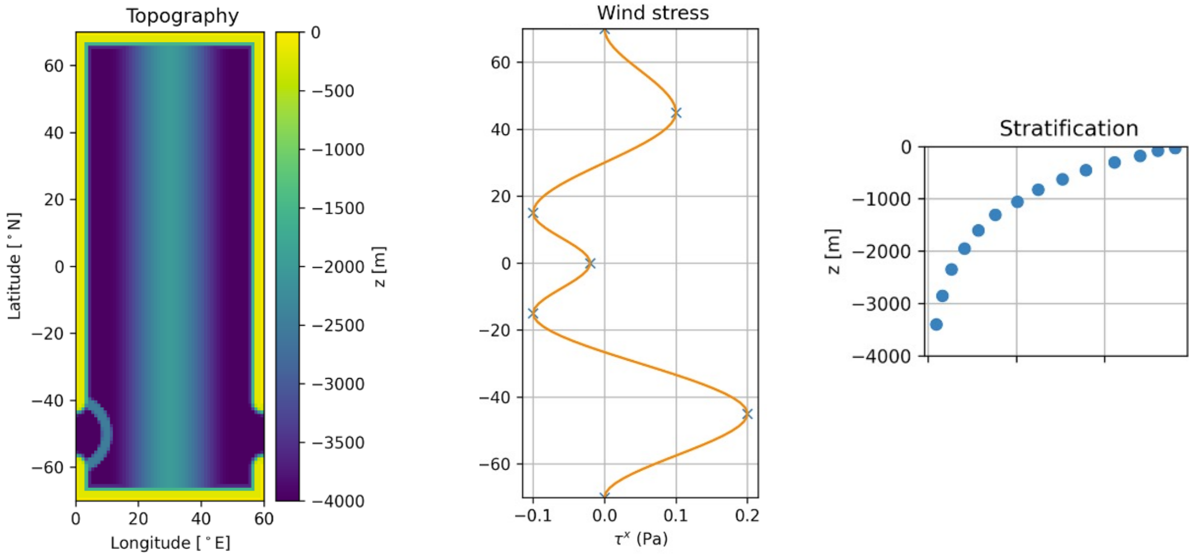
→ meridional ridge and continental slopes

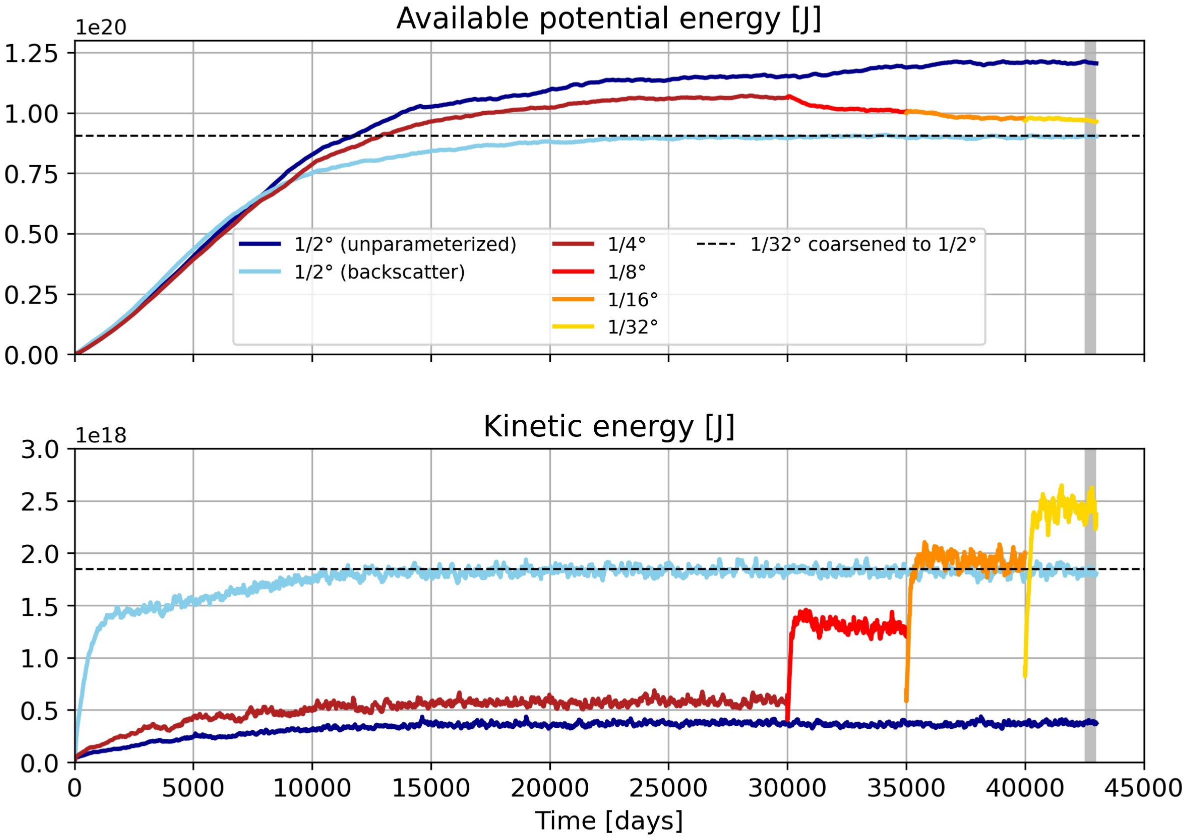
→ periodic southern channel

→ horizontal resolution:
  - ○ 1/2° (~50 km), “eddy-permitting”
  - ○ 1/32° (~3 km), “eddy-resolving”

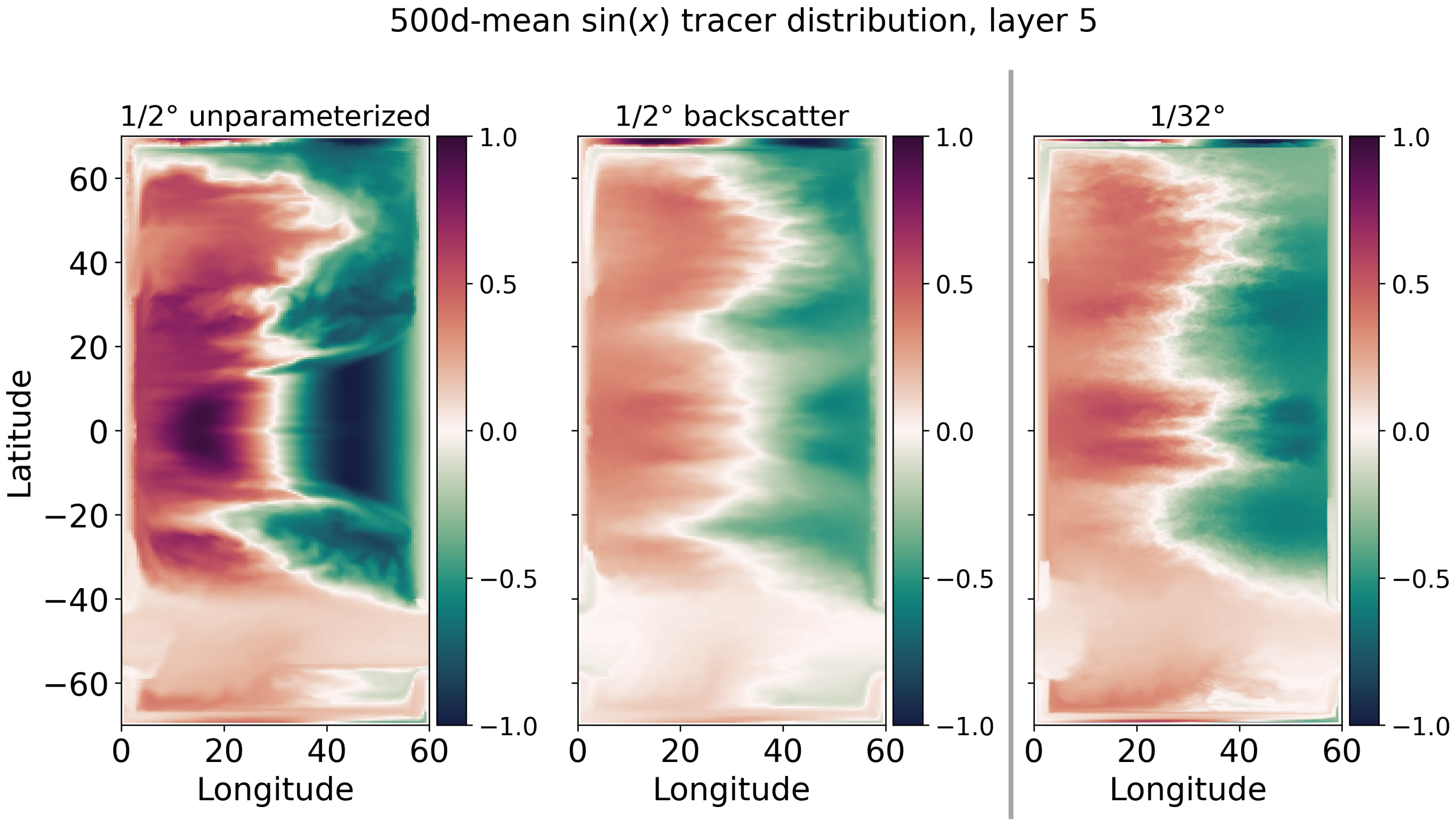
→ backscatter with **no GM** *(Yankovsky et al. 2024)*

→ passive tracers advected online
(forced-dissipative equilibrium)





Tracer distributions

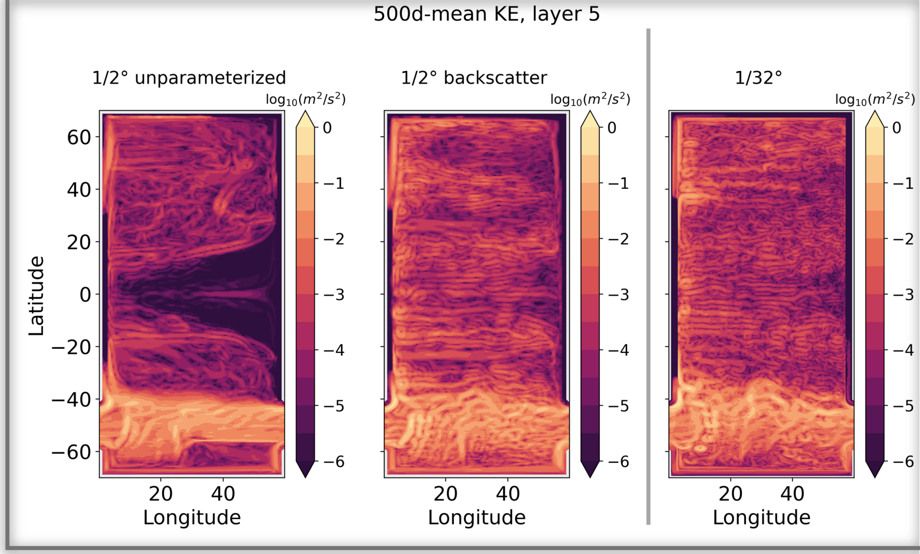


500d-mean sin(x) tracer distribution, layer 5

1/2° unparameterized

1/2° backscatter

1/32°



500d-mean KE, layer 5

1/2° unparameterized

1/2° backscatter

1/32°

Measuring eddy tracer fluxes

$$\widehat{c} \equiv \frac{\overline{hc}}{\overline{h}} \qquad c'' \equiv c - \widehat{c}$$

$$\begin{bmatrix} \widehat{u''c''} \\ \widehat{v''c''} \end{bmatrix} = - \begin{bmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{bmatrix} \begin{bmatrix} \partial_x \widehat{c} \\ \partial_y \widehat{c} \end{bmatrix}$$

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Method of Multiple Tracers (MMT)

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Method of Multiple Tracers (MMT)

- advect *n* tracers, $c_j(x, y, z, t), \quad j = 1, \dots, n$
- linearly restore to target fields: $\partial_t c_j + \cdots = -(c_j - c_j^*)/\tau_j$
- define 2 x 2 mixing tensor (at each *x, y, z*)

$$-K\nabla\widehat{c_j} \equiv \widehat{\boldsymbol{u}''c_j''}, \qquad K(x, y, z) \in \mathbb{R}^{2\times 2}$$

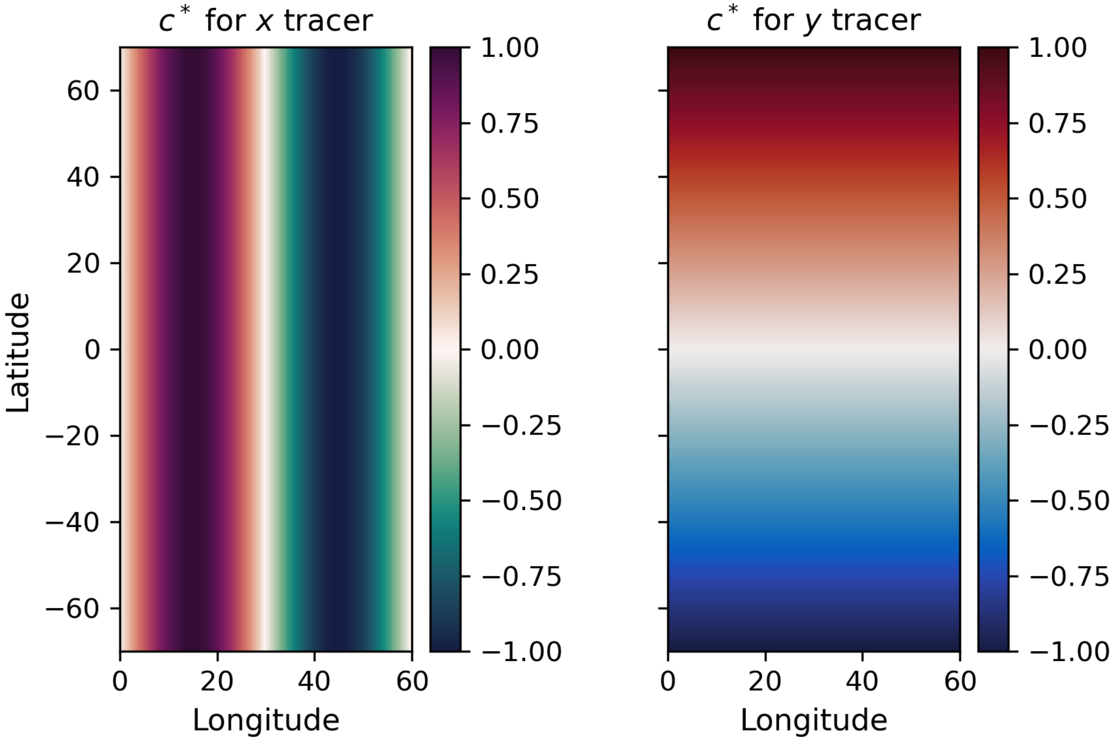
- write as over-determined matrix equation

$$-G^{\mathrm{T}}K^{\mathrm{T}} = F^{\mathrm{T}}, \qquad G, F \in \mathbb{R}^{2\times n}, \quad n > 2$$

- optimal solution from least squares

$$K_{\mathrm{lsq}}^{\mathrm{T}} = -(GG^{\mathrm{T}})^{-1}GF^{\mathrm{T}}$$

$$K \simeq K_{\mathrm{lsq}}$$



Tracer mixing

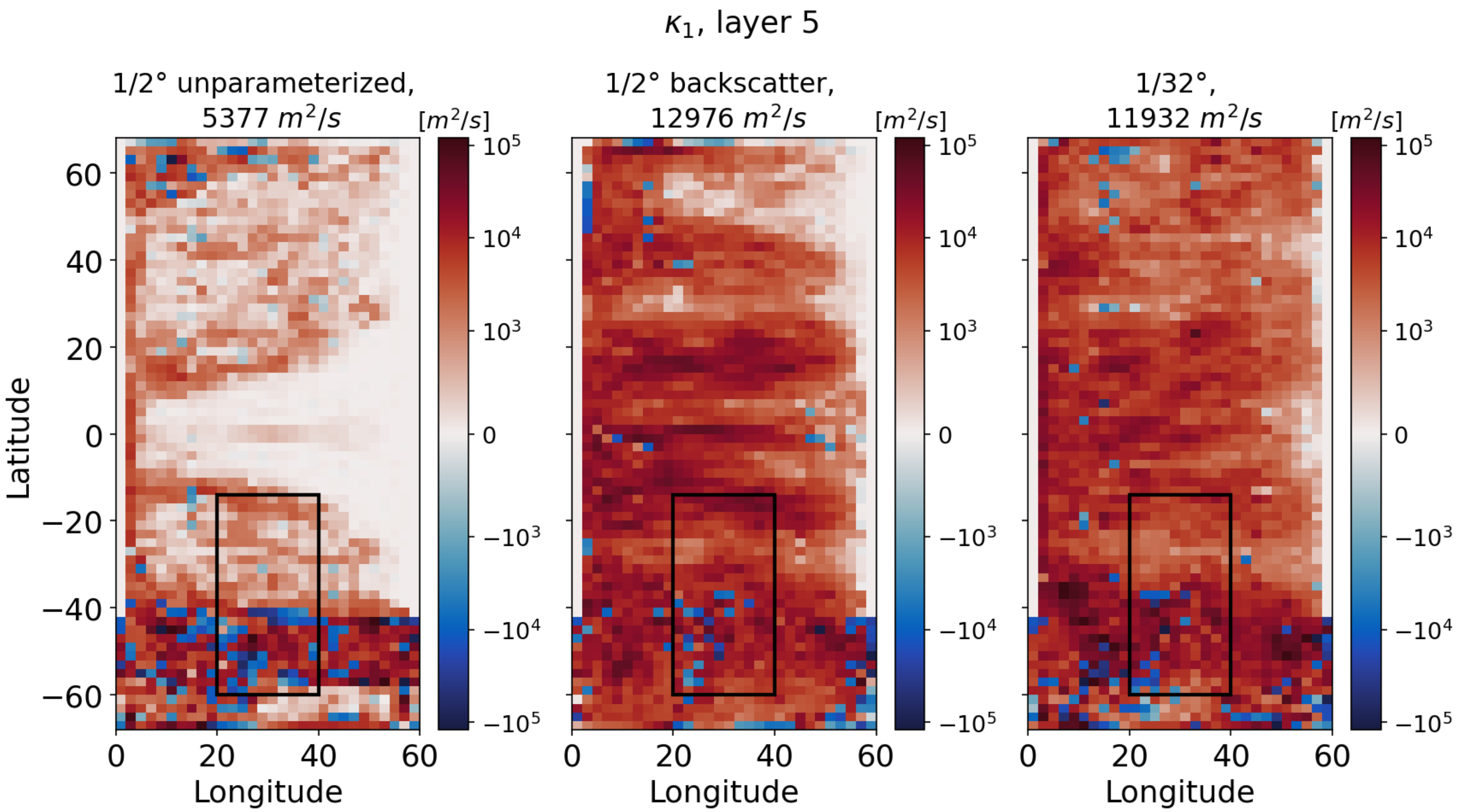
$$A = \frac{1}{2}(K - K^{\mathrm{T}})$$

$$S = \frac{1}{2}(K + K^{\mathrm{T}})$$

$$\implies \quad S = U \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix} U^{\mathrm{T}}$$

- Only symmetric part contributes to **irreversible mixing** (cf. Redi)

→ eigenvalues set magnitude



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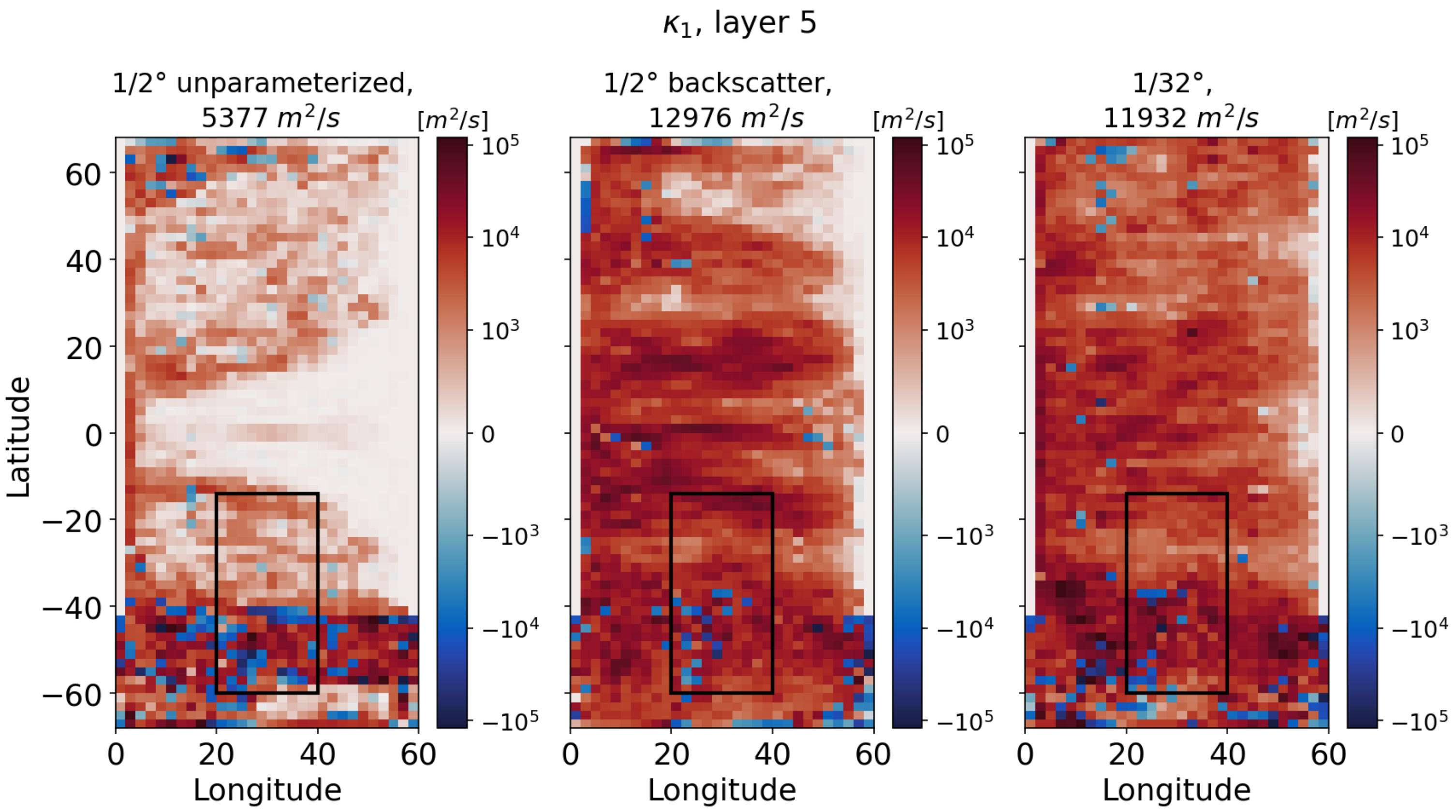


Global mean of eigenvalues (of positive values)

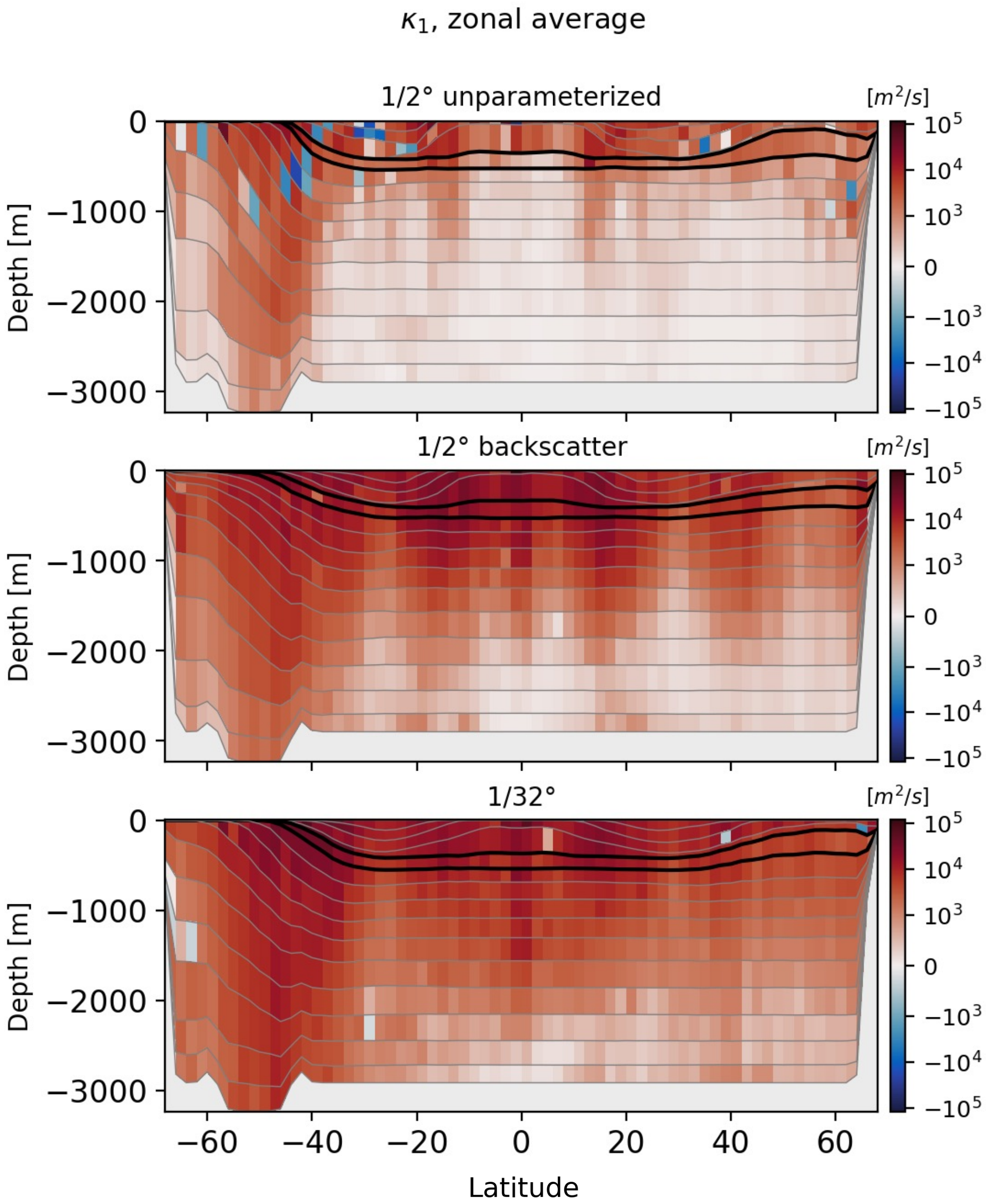
- 1/2° unparameterized: 2781 m^2s^{-1}
- 1/2° backscatter: 5341 m^2s^{-1}
- 1/32°: 5795 m^2s^{-1}

Tracer mixing

- similar zonal structure of mixing in backscatter and hi-res, but

→ weaker at depth in Northern part of domain

→ slightly lower magnitude in S.O.

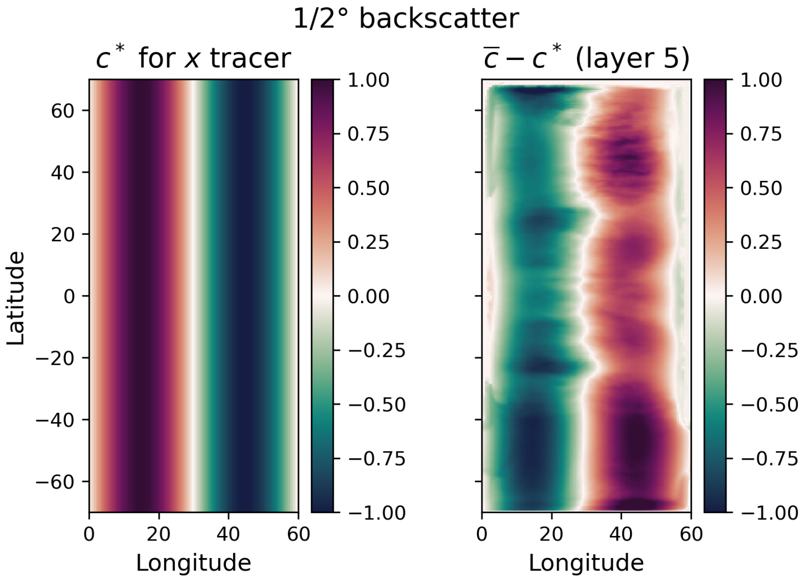


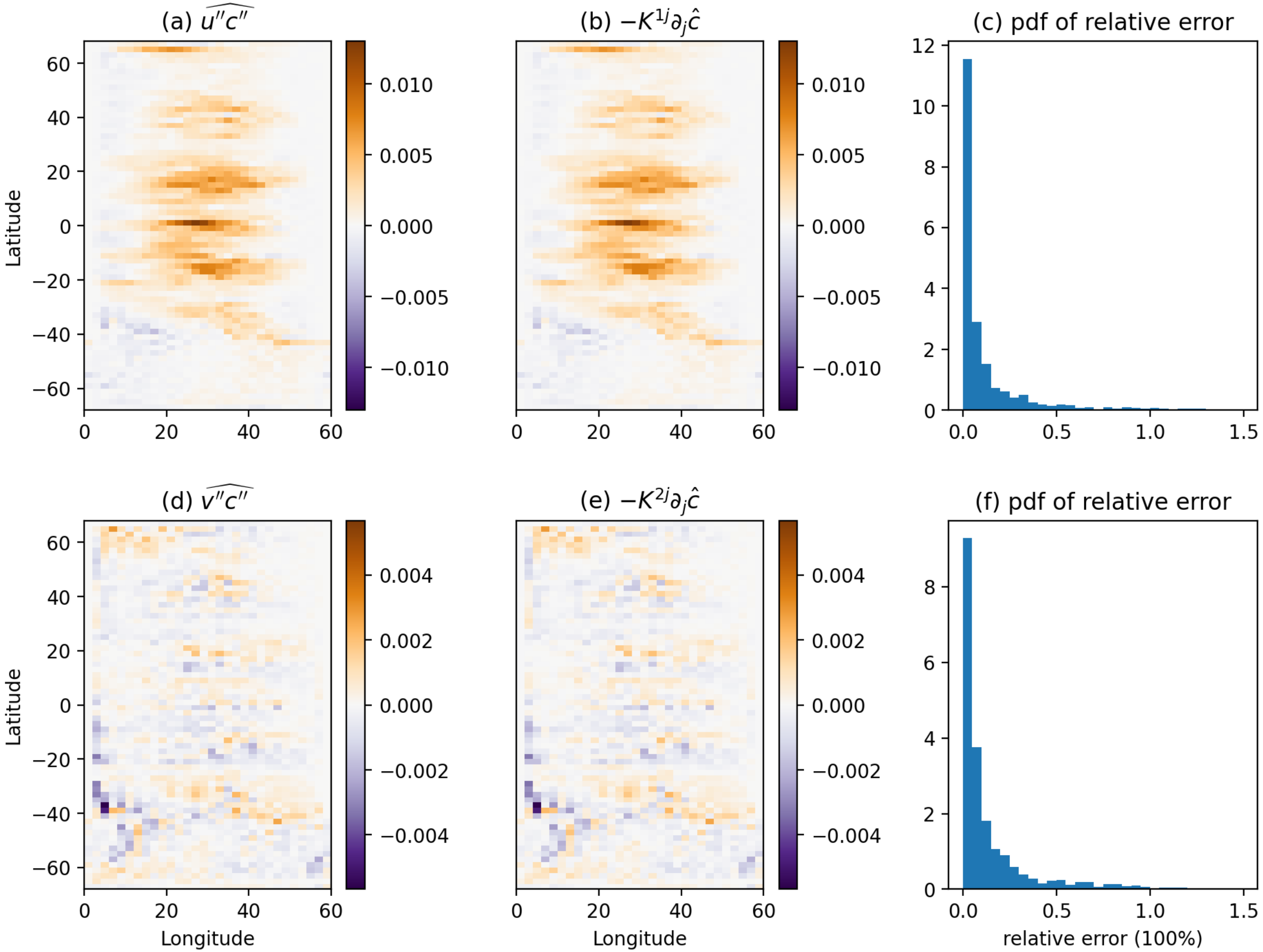
Flux-gradient reconstruction

Reconstruct tracer fluxes using full diffusivity tensor

→ using sin(x) tracer, *included* in the MMT inversion

→ verify fidelity of estimated diffusivities



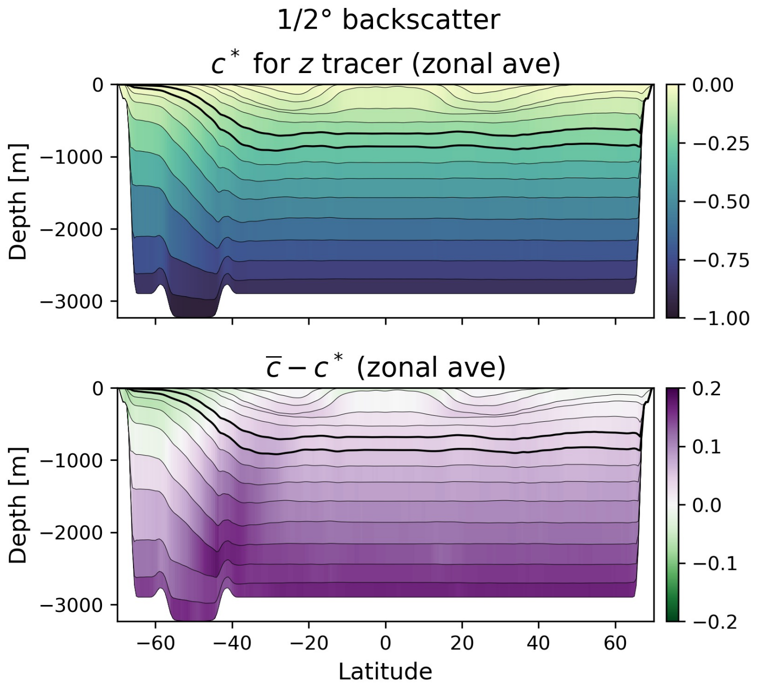


Flux-gradient reconstruction

Reconstruct tracer fluxes using full diffusivity tensor

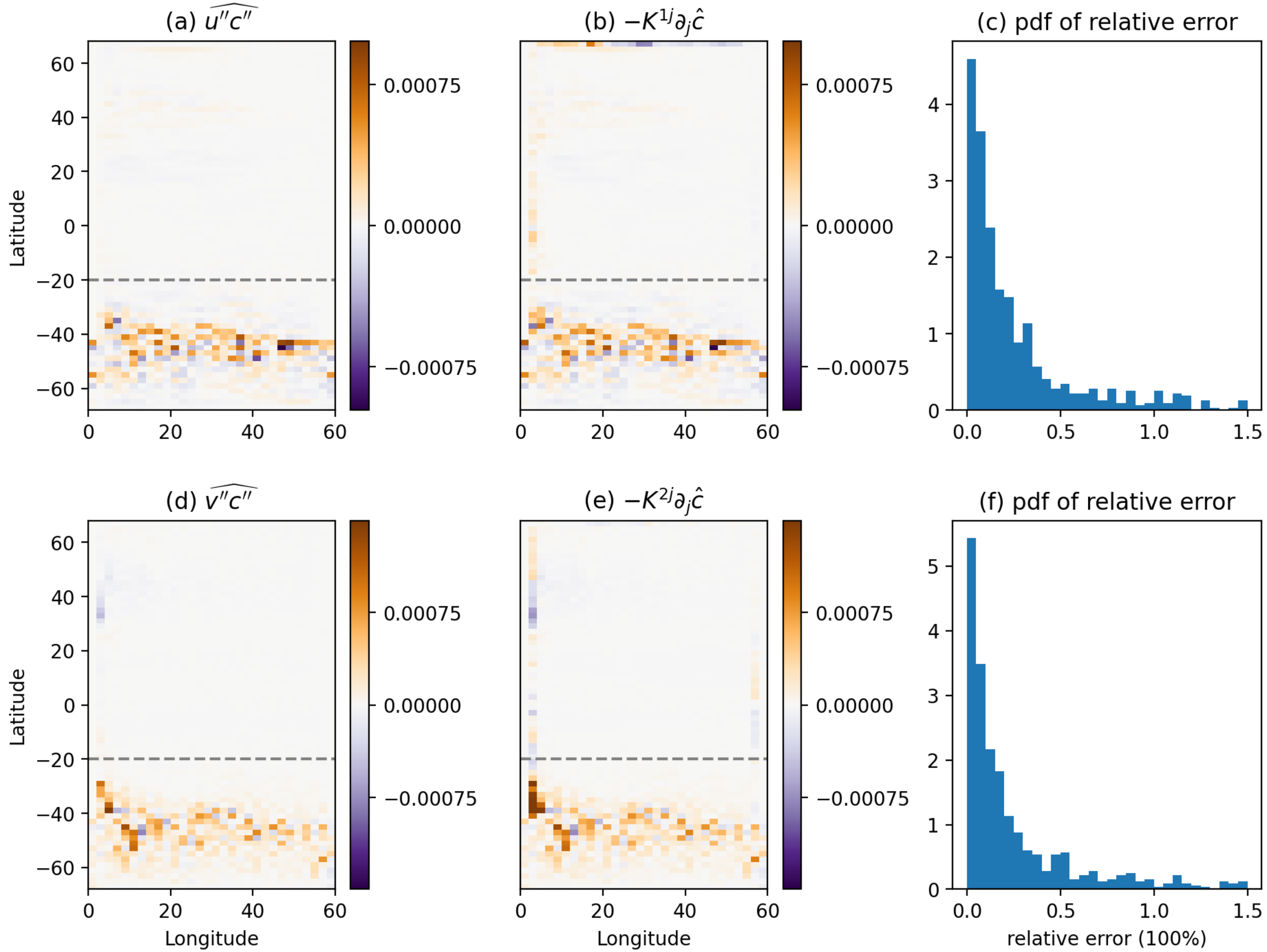
→ using *z* tracer, *not* included in the MMT inversion

→ fluxes/gradients confined to where isopycnals shoal


1/2° backscatter

c^* for z tracer (zonal ave)

$\overline{c} - c^*$ (zonal ave)

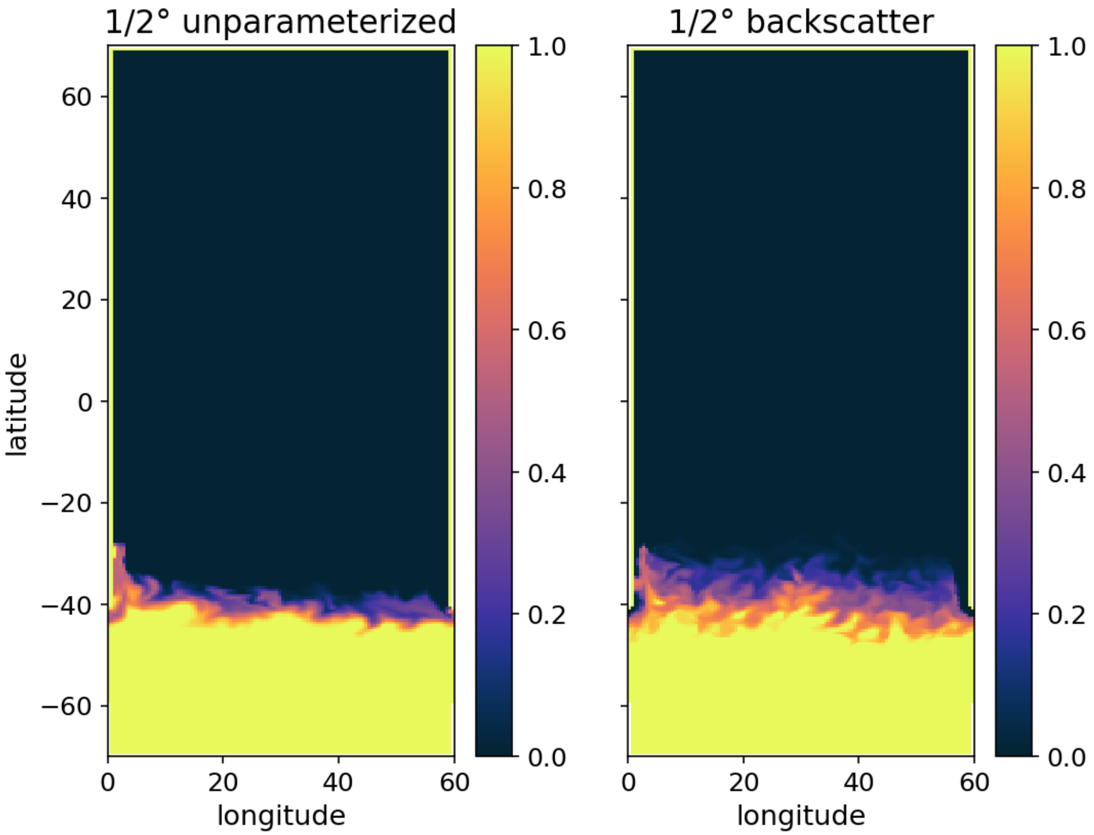


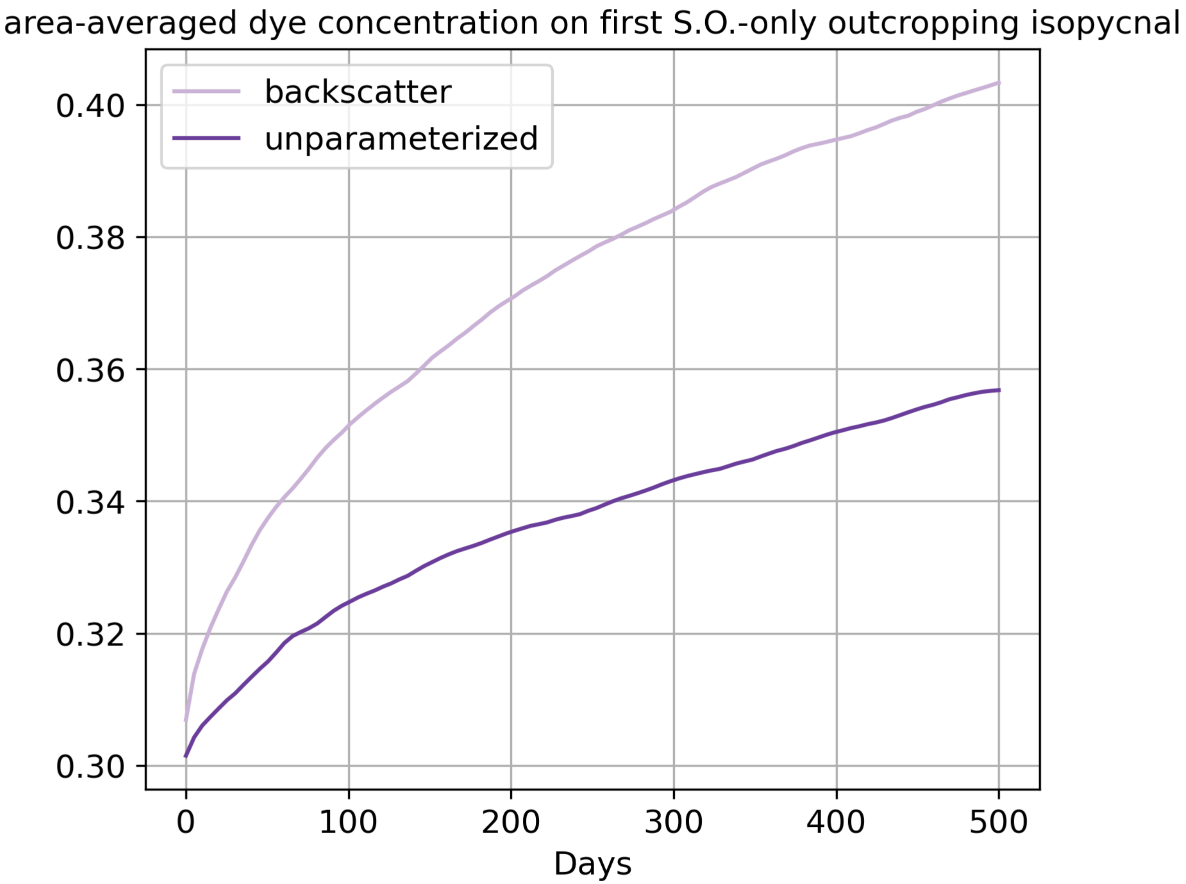
Surface dye tracer

- Surface dye tracer to assess **uptake**

→ Set to 1 if layer center < 100 m depth

→ Otherwise passively advected by flow





- ~13% **more uptake** in backscatter simulation

Summary and next steps

Summary

- Backscatter parameterization with **no GM or Redi** largely induces sufficient isopycnal mixing in 1/2° simulation to match hi-res truth
- Multiple tracers used to estimate spatial maps of isopycnal diffusivities
- More uptake of surface tracer in backscatter than unparameterized simulation

Next steps

- Currently examining 1/4° (unparameterized and backscatter)
- Other metrics to assess where backscatter is getting things right/wrong